Probability And Random Variables

① Find the value of k for a cts rv x whose density func. is given by

$f(x) = kx^2e^{-x}, x \geq 0.$

Ans: WKT $\int_{-\infty}^{\infty} f(x)dx = 1$

$k\int_0^{\infty} x^2e^{-x}dx = 1$

$k\left(-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right)_0^{\infty} = 1$

$2k = 1 \Rightarrow k = 1/2$

$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$

$u = x^2 \qquad dv = e^{-x}dx$

$u' = 2x \qquad v = \frac{e^{-x}}{-1}$

$u'' = 2 \qquad v_1 = \frac{e^{-x}}{(-1)^2}$

$v_2 = \frac{e^{-x}}{(-1)^3}$

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② Let A & B be two events such that $P(A)=0.5$, $P(B)=0.3$ & $P(A\cap B)=0.15$ compute $P(B|A)$ & $P(\bar{A}\cap B)$.

Ans: Given $P(A)=0.5$, $P(B)=0.3$, $P(A\cap B)=0.15$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{0.15}{0.5} = 0.3$$

$$P(\bar{A}\cap B) = P(B) - P(A\cap B) = 0.3 - 0.15 = 0.15$$

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③ Let A & B be 2 events such that $P(A)=1/3$, $P(B)=3/4$ & $P(A\cap B)=1/4$. Compute $P(A|B)$ & $P(\bar{A}\cap\bar{B})$.

Ans: $$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1/4}{3/4} = 1/3$$

$$P(\bar{A}\cap\bar{B}) = P(\overline{A\cup B}) = 1 - P(A\cup B) = 1 - [P(A)+P(B)-P(A\cap B)]$$

$$= 1 - [1/3 + 3/4 - 1/4] = 1/6$$

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④ A bag contains 8 white & 4 black balls. If 5 balls are drawn at random, what is the probability that 3 are white & 2 are black?

Ans: Total no. of balls $= 8+4 = 12$

$S = \{5 \text{ balls are taken out of } 12\}$

$$n(S) = 12C_5 = \frac{12\times11\times10\times9\times8}{1\times2\times3\times4\times5} = 792$$

$n(A) = 8C_3 \times 4C_2 = 336$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{336}{792} = 0.4242$$

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(5) Let $M_x(t) = \frac{1}{1-t}$, $|t|<1$, the mgf of rv X. Find $E(X)$ & $E(X^2)$.

Ans: Given $M_x(t) = \frac{1}{1-t} = (1-t)^{-1} = 1+t+t^2+\dots = 1+t+2\cdot\frac{t^2}{2!}+\dots$

$E(X) =$ coeff. of t in $M_x(t) = 1$

$E(X^2) =$ " " $\frac{t^2}{2!}$ " " $= 2$

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(6) A rv X has probability mass func. $P(X=x) = \frac{x}{10}$, $x = 1,2,3,4$. Find the cumulative distribution func. $F(x)$ of X.

Ans:

x :	1	2	3	4
$P(x)$:	$1/10$	$2/10$	$3/10$	$4/10$

x	$P(x)$	$F(x) = P(X \le x)$
1	$1/10$	$F(1) = P(1) = 1/10$
2	$2/10$	$F(2) = P(1)+P(2) = 1/10 + 2/10 = 3/10$
3	$3/10$	$F(3) = P(1)+P(2)+P(3) = 6/10$
4	$4/10$	$F(4) = P(1)+P(2)+P(3)+P(4) = 1$

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(7) The rv X has pmf $P(X=x) = \begin{cases} c/x, & x=1,2,3 \\ 0, & \text{otherwise} \end{cases}$. Obtain (i) the value of c (ii) $P(X \ge 2)$.

Ans:

x :	1	2	3
$P(x)$:	$c/1$	$c/2$	$c/3$

(i) $\sum_{x=1}^{3} P(x) = 1$

$\frac{c}{1} + \frac{c}{2} + \frac{c}{3} = 1$

$c\left(\frac{11}{6}\right) = 1$

$c = \frac{6}{11}$

(ii) $P(X \ge 2) = P(2) + P(3)$

$= c/2 + c/3 = \frac{5c}{6} = \frac{5}{6} \times \frac{6}{11} = \frac{5}{11}$

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(8) S.T. for any events A & B in S, $P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})$

Ans: RHS $= P(B|A)P(A) + P(B|\bar{A})P(\bar{A})$

$= \frac{P(A\cap B)}{P(A)} P(A) + \frac{P(\bar{A}\cap B)}{P(\bar{A})} P(\bar{A})$

$= P(A\cap B) + P(\bar{A}\cap B) = P(A\cap B) + P(B) - P(A\cap B) = P(B)$

$=$ LHS

(9) The pdf of the rv X is given by $f(x)=\begin{cases} k(1-x^2), & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$. Find the value of k.

Ans: WKT $\int_{-\infty}^{\infty} f(x)dx = 1$

$$k\int_0^1 (1-x^2)\,dx = 1$$

$$k\left(x-\frac{x^3}{3}\right)_0^1 = 1 \Rightarrow k\left(1-\frac{1}{3}\right)=1 \Rightarrow k=\frac{3}{2}$$

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(10) If A & B are mutually exclusive events $P(A)=0.29$ & $P(B)=0.43$ then find $P(\bar{A})$ & $P(A\cup B)$.

Ans: $P(\bar{A}) = 1-P(A) = 1-0.29 = 0.71$

$P(A\cup B) = P(A)+P(B) = 0.29+0.43 = 0.72$

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(11) A coin is tossed 3 times. If X denotes the number of heads obtained, find the probability distribution of X.

Ans: Probability distribution of X: X – number of heads obtained

x:	0	1	2	3
$P(x)$:	$1/8$	$3/8$	$3/8$	$1/8$

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(12) State memory less property & which continuous & discrete distributions follow this property?

Ans: Memoryless property:

If X is a random variable, then $P(X>m+n / X>m) = P(X>n)$ for any $m,n>0$.

In continuous distribution, exponential distribution follows memoryless property & in discrete distribution, geometric distribution follows memoryless property.

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(13) A random variable X has a uniform distribution over $(-3,3)$. Compute $P(|X-2|<2)$.

Ans: Given X is a uniform random variable in the interval $(-3,3)$.

(ie) $a=-3, b=3$.

Pdf: $f(x) = \frac{1}{b-a} = \frac{1}{3+3} = \frac{1}{6}, \ -3 < x < 3$

$P(|x-2| < 2) = P(-2 < x-2 < 2) = P(-2+2 < x < 2+2)$

$= P(0 < x < 4) = \int_0^3 f(x)\,dx = \int_0^3 \frac{1}{6}\,dx$

$= \frac{1}{6}(x)_0^3 = \frac{1}{6}(3-0) = \frac{1}{2}$

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(14) If the pdf of x is given by $f(x) = \begin{cases} 2(1-x), & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$ then show that $E(x^r) = \frac{2}{(r+1)(r+2)}$.

Ans: Given $f(x) = \begin{cases} 2(1-x), & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

$E[x^r] = \int_{-\infty}^{\infty} x^r f(x)\,dx = \int_0^1 x^r\, 2(1-x)\,dx = 2\int_0^1 (x^r - x^{r+1})\,dx$

$= 2\left[\frac{x^{r+1}}{r+1} - \frac{x^{r+2}}{r+2}\right]_0^1 = 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right]$

$= 2\left[\frac{r+2-r-1}{(r+1)(r+2)}\right] = 2\left[\frac{1}{(r+1)(r+2)}\right] = \frac{2}{(r+1)(r+2)}$

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(15) Find k, if the pdf of x is

x :	-1	0	1	2	3
$P(x=x)$:	2K	3K	4K	$6K^2$	$4K^2$

Ans: WKT $\sum P(x) = 1$

$2K + 3K + 4K + 6K^2 + 4K^2 = 1$

$9K + 10K^2 = 1$

$10K^2 + 9K - 1 = 0$

×	+
−10	9
−1	+10
10k−1	10k+10
	k+1

$(10K-1)(K+1) = 0$

$10K = 1 \qquad K+1 = 0$

$K = 1/10 \qquad K = -1$

$\therefore K = 1/10$

TWO DIMENSIONAL RANDOM VARIABLES

① Let X & Y be two independent rvs with $Var(X)=9$ & $Var(Y)=3$. Find $Var(4X-2Y+6)$.

Sol: $Var(4X-2Y+6) = 4^2 Var(X) + (-2)^2 Var(Y) = (16\times 9) + (4\times 3) = 156$

② The joint pdf of (X,Y) is $f(x,y) = \begin{cases} 4xy, & 0\le x\le 1, 0\le y\le 1/2 \\ 0, & \text{otherwise} \end{cases}$. Calculate $P(X\le 2Y)$.

Sol: $$P(X\le 2Y) = \int_0^{1/2}\int_0^{2y} 4xy\,dx\,dy$$

$$= 4\int_0^{1/2} y\left(\frac{x^2}{2}\right)_0^{2y} dy = 2\int_0^{1/2} y(4y^2)\,dy = 8\left(\frac{y^4}{4}\right)_0^{1/2} = 2\left(1/16\right) = 1/8$$

③ Define covariance & coefficient of correlation between 2 rvs x & y.

Sol: $Cov(X,Y) = E(XY) - E(X)E(Y)$

$$r(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$\sigma_X = \sqrt{Var X}$ & $\sigma_Y = \sqrt{Var Y}$, $Var(X) = E(X^2) - [E(X)]^2$, $Var(Y) = E(Y^2) - [E(Y)]^2$

④ The joint pdf of a bivariate rv (X,Y) is given by $f(x,y) = \begin{cases} k, & 0 < y \le x < 1 \\ 0, & \text{otherwise} \end{cases}$ where k is a constant. Determine the value of k.

Sol: WKT $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$$k\int_0^1\int_0^x dy\,dx = 1 \Rightarrow k\int_0^1 (y)_0^x\,dx = 1 \Rightarrow k\int_0^1 x\,dx = 1$$

$$k\left(\frac{x^2}{2}\right)_0^1 = 1 \Rightarrow k\left(1/2\right) = 1 \Rightarrow k = 2$$

⑤ P.T. the correlation coeff. ρ_{xy} of the rvs X & Y takes value in the range -1 & 1.

Sol: WKT $r = \frac{\rho}{\sigma_x \sigma_y}$

$$\rho^2 = \left[\frac{\sum(x-\bar{x})(y-\bar{y})}{n}\right]^2 = \left(\frac{\sum XY}{n}\right)^2$$

$$\sigma_x^2\sigma_y^2 = \frac{\sum X^2 \sum Y^2}{n^2}$$

$$\left(\sum (xy)\right)^2 \le \left(\sum x^2\right)\left(\sum y^2\right)$$

$$\frac{\left(\sum xy\right)^2}{n^2} \le \frac{\sum x^2 \sum y^2}{n^2}$$

$$\rho^2 \le \sigma_x^2 \sigma_y^2$$

$$(r\sigma_x\sigma_y)^2 \le \sigma_x^2\sigma_y^2$$

$$r^2\sigma_x^2\sigma_y^2 \le \sigma_x^2\sigma_y^2$$

$$r^2 \le 1 \Rightarrow |r| \le 1 \Rightarrow -1 \le r \le 1$$

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(6) Let (x, y) be a two-dimensional rv. Define covariance of (x, y). If x & y are independent. What will be the covariance of (x, y)?

Sol: $\text{Cov}(x, y) = E(xy) - E(x)E(y)$

$= E(x)E(y) - E(x)E(y)$ ($\because x$ & y are independent)

$= 0$

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(7) Can $y = 5 + 2.8x$ & $x = 3 - 0.5y$ be the estimated regression eqn. of y on x respectively explain your answer.

Sol: $b_{yx} = 2.8$, $b_{xy} = -0.5$

$r = \pm\sqrt{b_{xy} \cdot b_{yx}}$ = imaginary

$\therefore$ They cannot be estimated regression eqns.

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(8) The joint pmf of a 2-dimensional rv (x, y) is given by $P(x, y) = k(2x + y)$, $x = 1, 2$ & $y = 1, 2$, where k is a constant. Find the value of k.

Sol:

$x \backslash y$	1	2
1	3k	4k
2	5k	6k

$18k = 1$

$k = 1/18$

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(9) If the joint pdf of (x, y) is $f(x, y) = \begin{cases} 1/4, & 0 \le x, y \le 2 \\ 0, & \text{otherwise} \end{cases}$, find $P(x + y \le 1)$.

Sol: $P(x + y \le 1) = \int_0^1 \int_0^{1-y} f(x, y)\,dx\,dy = \frac{1}{4}\int_0^1 \int_0^{1-y} dx\,dy$

$= \frac{1}{4}\int_0^1 (x)_0^{1-y} dy = \frac{1}{4}\int_0^1 (1 - y)\,dy = \frac{1}{4}\left(y - \frac{y^2}{2}\right)_0^1$

$= \frac{1}{4}\left(1 - \frac{1}{2}\right) = \frac{1}{8}$

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(10) Determine the value of the constant c if the joint density func. of 2 discrete rvs X & Y is given by $p(m,n) = cmn$, $m = 1,2,3$ & $n = 1,2,3$.

Sol: Given $p(m,n) = cmn$, $m = 1,2,3$ & $n = 1,2,3$

n\m	1	2	3
1	c	2c	3c
2	2c	4c	6c
3	3c	6c	9c

$36c = 1$

$c = 1/36$

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(11) Determine the value of k if $f(x,y) = \begin{cases} kxe^{-y} ; & 0 < x < 2, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$ is a joint pdf of two dimensional rv (x, y).

Ans: Given $f(x,y) = \begin{cases} kxe^{-y} ; & 0 < x < 2, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$

WKT $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$

$\int_0^2 \int_0^{\infty} kxe^{-y}\, dy\, dx = 1 \Rightarrow k \int_0^2 x \left(\frac{e^{-y}}{-1}\right)_0^{\infty} dx = 1$

$-k \int_0^2 x(0-1)\, dx = 1 \Rightarrow k \int_0^2 x\, dx = 1 \Rightarrow k\left(\frac{x^2}{2}\right)_0^2 = 1$

$\frac{k}{2}(4-0) = 1 \Rightarrow 2k = 1 \Rightarrow \boxed{k = 1/2}$

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(12) When will the 2 regression lines be (a) at right angles, (b) coincident?

Ans: When $r = 0$, $\tan\theta = \infty \Rightarrow \theta = \pi/2$ & so the regression lines are perpendicular.

When $r = 1$ or -1, $\theta = 0$ & so the regression lines are coincide.

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(13) The joint pdf of (x, y) is $f(x,y) = \begin{cases} 4xy, & 0 \leq x \leq 1, 0 \leq y \leq 1. \\ 0, & \text{otherwise} \end{cases}$ Calculate $P(x \leq 2y)$.

Ans: Given $f(x,y) = \begin{cases} 4xy, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$

$$P(X \le 2Y) = 4\int_0^1 \int_0^{2y} xy\,dx\,dy = 4\int_0^1 \left(\frac{x^2}{2}\right)_0^{2y} y\,dy$$

$$= \frac{4}{2}\int_0^1 (4y^2)\,y\,dy = 2\int_0^1 4y^3\,dy = 8\left(\frac{y^4}{4}\right)_0^1$$

$$= 2$$

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(14) State the central limit theorem for independent & identically distributed random variables.

Ans: If $X_1, X_2, \ldots, X_n$ be a sequence of independent identically distributed random variables with $E[X_i] = \mu$ & $Var[X_i] = \sigma^2$, $i = 1, 2, \ldots, n$ & if $S_n = X_1 + X_2 + \cdots + X_n$, then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ & variance $n\sigma^2$ as $n \to \infty$.

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(15) If $f(x,y) = e^{-(x+y)}$, $x \ge 0, y \ge 0$, is the joint pdf of (X, Y), find $P(X + Y \le 1)$.

Ans: Given $f(x,y) = e^{-(x+y)}$, $x \ge 0, y \ge 0$

$$P(X+Y \le 1) = \int_0^1 \int_0^{1-y} e^{-(x+y)}\,dx\,dy$$

$$= \int_0^1 \left(\frac{e^{-x}}{-1}\right)_0^{1-y} e^{-y}\,dy = -\int_0^1 \left(e^{-(1-y)} - 1\right) e^{-y}\,dy$$

$$= -\int_0^1 \left(e^{-1} - e^{-y}\right) dy = -\left(ye^{-1} - \frac{e^{-y}}{-1}\right)_0^1$$

$$= -\left(e^{-1} + e^{-1} - 1\right) = -\left(2e^{-1} - 1\right) = 1 - 2e^{-1}$$

ANALYTIC FUNCTIONS

① Is the function $f(z) = \bar{z}$ analytic? (or)
Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

Ans: Given $f(z) = \bar{z}$ — ①

Let $z = x+iy$, $f(z) = u+iv$

① $\Rightarrow u+iv = \overline{x+iy} = x - iy$

$u = x$, $v = -y$

$u_x = 1 \qquad v_x = 0$

$u_y = 0 \qquad v_y = -1$

$\therefore u_x \neq v_y \Rightarrow$ Cauchy-Riemann eqns/. are not satisfied.
Hence $f(z) = \bar{z}$ is not analytic (nowhere differentiable)

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② Is the function $f(z) = |z|^2$ analytic? (or)
Give an example of a function which is differentiable at a point but not analytic at that point.

Ans: Given $f(z) = |z|^2$

$z = x+iy$, $f(z) = u+iv$

$u+iv = |x+iy|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2$

$u = x^2+y^2$, $v = 0$

$u_x = 2x \qquad v_x = 0$

$u_y = 2y \qquad v_y = 0$

$u_x \neq v_y$, $u_y \neq -v_x$ except at $x=0$, $y=0$ only.
$\therefore f(z) = |z|^2$ is not analytic but differentiable at $(0,0)$.

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③ Give an example of a function where u & v are harmonic but $u+iv$ is not analytic.

Ans: Take $u = x^2 - y^2$ & $v = \dfrac{-y}{x^2+y^2}$

u & v are harmonic.

$u_x = 2x$, $v_y \neq 2x$

$\therefore u+iv$ is not analytic.

4. Find the critical points of $w^2 = (z-\alpha)(z-\beta)$.

Ans: Critical points at $\frac{dw}{dz} = 0$ & $\frac{dz}{dw} = 0$.

Given $w^2 = (z-\alpha)(z-\beta)$

Diff /. w.r.t. z,

$$2w\frac{dw}{dz} = (z-\alpha) + (z-\beta)$$

$$\frac{dw}{dz} = \frac{(z-\alpha)+(z-\beta)}{2w} \quad \text{--- (1)}$$

$$\Rightarrow \frac{dz}{dw} = \frac{2w}{(z-\alpha)+(z-\beta)} \quad \text{--- (2)}$$

Put $\frac{dw}{dz} = 0 \Rightarrow (1) \Rightarrow \frac{2z-\alpha-\beta}{2w} = 0$

$$\Rightarrow 2z - \alpha - \beta = 0$$

$$\Rightarrow 2z = \alpha + \beta$$

$$\Rightarrow z = \frac{\alpha+\beta}{2}$$

Put $\frac{dz}{dw} = 0$

(2) $\Rightarrow \frac{2w}{(z-\alpha)+(z-\beta)} = 0$

$$\Rightarrow w = 0$$

$$\Rightarrow \sqrt{(z-\alpha)(z-\beta)} = 0 \qquad (\because w^2 = (z-\alpha)(z-\beta))$$

$$\Rightarrow z = \alpha, \beta$$

$\therefore$ Critical points are $\frac{\alpha+\beta}{2}, \alpha, \beta$.

5. Find the value of m if $u = 2x^2 - my^2 + 3x$ is harmonic.

Ans: Since u is harmonic,

To prove: $u_{xx} + u_{yy} = 0$

Given $u = 2x^2 - my^2 + 3x$

$u_x = 4x + 3 \qquad u_y = -2my$

$u_{xx} = 4 \qquad u_{yy} = -2m$

$u_{xx} + u_{yy} = 0 \Rightarrow 4 - 2m = 0 \Rightarrow \boxed{m = 2}$

(6) Find the image of a circle $|z|=3$ under the transformation $w=2z$.

Ans: Let $z=x+iy$, $w=u+iv$

Given $w=2z \Rightarrow z=\frac{w}{2}$

$$x+iy=\frac{u+iv}{2}=\frac{u}{2}+i\frac{v}{2}$$

$$x=\frac{u}{2}, \quad y=\frac{v}{2}$$

Given $|z|=3$

$$|x+iy|=3$$

$$\sqrt{x^2+y^2}=3$$

$$x^2+y^2=3^2$$

$$\left(\frac{u}{2}\right)^2+\left(\frac{v}{2}\right)^2=3^2$$

$$\frac{u^2}{4}+\frac{v^2}{4}=9 \Rightarrow u^2+v^2=36$$

It is a circle with centre at $(0,0)$ & radius 6.

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(7) If c is the circle $|z|=3$ & if $g(z_0)=\int_c \frac{2z^2-z-2}{z-z_0}dz$ then find $g(2)$.

Ans: $g(2)=\int_c \frac{2z^2-z-2}{z-2}dz$

By Cauchy's integral theorem, $\int_c \frac{f(z)}{z-a}dz=2\pi i f(a)$

Here $a=2$.

$$f(z)=2z^2-z-2$$

$$f(2)=2(4)-2-2=4$$

$$\therefore \int_c \frac{2z^2-z-2}{z-2}dz=2\pi i f(2)=2\pi i\times 4=8\pi i$$

(ie), $g(2)=8\pi i$.

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(8) Find the value of $\int_c \frac{3z^2+7z+1}{z+1}\,dz$ if c is $|z| = 1/2$.

Ans: Given fun. is not analytic at $z=-1$.

Given $|z| = 1/2$

$z=-1$, $|-1| = 1 > 1/2$, lies outside the circle.

$\therefore$ By Cauchy's integral thm.,

$$\int_c \frac{3z^2+7z+1}{z+1}\,dz = 0.$$

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(9) The real part of an analytic fun. $f(z)$ is constant, prove that $f(z)$ is a constant fun..

Ans:

Given $f(z) = u+iv$ is analytic & u is constant.

$\Rightarrow u_x = 0$ & $u_y = 0$

Since $f(z)$ is analytic, $u_x = v_y$ & $v_x = -u_y$

$\Rightarrow v_y = 0$ & $v_x = 0$

$\Rightarrow v$ is independent of x & y.

$\Rightarrow v = \text{constant}$

$\therefore f(z) = \text{constant}.$

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(10) Find the critical points of the transformation $w = z^2 + \frac{1}{z^2}$.

Ans: Critical point at $\frac{dw}{dz} = 0$ & $\frac{dz}{dw} = 0$

Given $w = z^2 + \frac{1}{z^2}$

$$\frac{dw}{dz} = 2z - \frac{2}{z^3}$$

$$\frac{dw}{dz} = \frac{2z^4-2}{z^3}$$

$$\frac{dz}{dw} = \frac{z^3}{2z^4-2}$$

$$\frac{dw}{dz} = 0 \Rightarrow \frac{2z^4-2}{z^3} = 0 \Rightarrow 2z^4 = 2 \Rightarrow z^4 = 1 \Rightarrow z = 1, -1, i, -i$$

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$$\frac{dz}{dw}=0 \Rightarrow \frac{z^3}{2z^4-2}=0$$

$$\Rightarrow z^3=0 \Rightarrow z=0$$

$\therefore$ Critical points are $z=0, 1, -1, i, -i$.

(11) Find the invariant points of a function $f(z)=\frac{z^3+7z}{7-6zi}$.

Ans: Invariant points (or) fixed points is $z=f(z)$.

Given $f(z)=\frac{z^3+7z}{7-6zi}$

$$\Rightarrow z=\frac{z^3+7z}{7-6zi}$$

$$7z-6z^2i=z^3+7z$$

$$z^3+6z^2i=0$$

$$z^2(z+6i)=0$$

$$z=0, \quad z=-6i$$

(12) Examine whether the fun. $u=xy^2$ can be a real part of an analytic fun..

Ans: WKT, Real part of an analytic fun. satisfies Laplace eqn. $u_{xx}+u_{yy}=0$.

Given $u=xy^2$

$$u_x=y^2 \qquad u_{xx}=0$$

$$u_y=2xy \qquad u_{yy}=2x$$

$$u_{xx}+u_{yy}=0+2x=2x\neq 0$$

$\therefore u=xy^2$ cannot be a real part of an analytic fun..

(13) If $f(z)=r^2(\cos 2\theta+i\sin p\theta)$ is analytic, then find the value of p.

Ans: In polar form, C-R eqn. is

$$u_r=\frac{1}{r}v_\theta, \quad v_r=\frac{-1}{r}u_\theta \quad \text{——} (1)$$

Here $u = r^2 \cos 2\theta$ $\quad v = r^2 \sin p\theta$

$u_r = 2r \cos 2\theta$ $\quad v_r = 2r \sin p\theta$

$u_\theta = -2r^2 \sin 2\theta$ $\quad v_\theta = p r^2 \cos p\theta$

(1) $\Rightarrow$ $2r\cos 2\theta = \frac{1}{r} p r^2 \cos p\theta$

$2r\cos 2\theta = pr\cos p\theta$

$\Rightarrow \boxed{p = 2}$

COMPLEX INTEGRATION

① If c is the circle $|z|=3$ & $g(z_0) = \int_C \frac{2z^2-z-2}{z-z_0} dz$ then find $g(2)$.

Sol: Let $g(2) = \int_c \frac{2z^2-z-2}{z-2} dz$

Let $f(z) = 2z^2-z-2$ & $z_0 = 2$ lies inside c.

By Cauchy integral formula,

$$\int_c \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$$

$$\int_c \frac{2z^2-z-2}{z-2} dz = 2\pi i f(2) = 2\pi i(4) = 8\pi i$$

② Find the value of $\int_c \frac{3z^2+7z+1}{z+1} dz$ if c is $|z| = 1/2$.

Sol: $z_0 = -1$ lies outside c.

By Cauchy integral theorem, $\int_c f(z) dz = 0$.

$$\therefore \int_c \frac{3z^2+7z+1}{z+1} dz = 0$$

③ Evaluate $\int_c \frac{e^z}{z-2} dz$ if c is unit circle with centre as origin.

Sol: Given circle c is $|z|=1$

$z_0 = 2$ lies outside the circle $|z|=1$.

By Cauchy integral theorem, $\int_c f(z) dz = 0$

$$\therefore \int_c \frac{e^z}{z-2} dz = 0$$

④ Determine the residue of $f(z) = \frac{z+1}{(z-1)(z+2)} dz$ at $z=1$.

Sol: $$\operatorname{Res}_{z=1} f(z) = \lim_{z\to 1} (z-1) \frac{z+1}{(z-1)(z+2)} = \lim_{z \to 1} \frac{z+1}{z+2}$$

$$= \frac{1+1}{1+2} = \frac{2}{3}$$

⑤ Expand $f(z)=\frac{1}{z^2}$ as a Taylor series about the point $z=2$.

Sol: Taylor series about the point $z=a$ is

$$f(z)=f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+\cdots$$

At $z=2$, $f(z)=f(2)+(z-2)f'(2)+\frac{(z-2)^2}{2!}f''(2)+\cdots$

$$f(z)=\frac{1}{z^2} \Rightarrow f(2)=\frac{1}{4}$$

$$f'(z)=\frac{-2}{z^3} \Rightarrow f'(2)=-\frac{2}{8}=\frac{-1}{4}$$

$$f''(z)=\frac{6}{z^4} \Rightarrow f''(2)=\frac{6}{16}=\frac{3}{8}$$

$$\therefore f(z)=\frac{1}{4}+(z-2)\left(\frac{-1}{4}\right)+\frac{(z-2)^2}{2}\left(\frac{3}{8}\right)+\cdots$$

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⑥ Evaluate the residue of $f(z)=\tan z$ at its singularities.

Sol: Let $f(z)=\frac{\sin z}{\cos z}=\frac{\phi(z)}{\psi(z)}$

Singularities are $z=\pm(2n-1)\frac{\pi}{2}$

$$\operatorname*{Res}_{z=a} f(z)=\lim_{z\to a}\frac{\phi(z)}{\psi'(z)}$$

$$\operatorname*{Res}_{z=\pm(2n-1)\pi/2} f(z)=\lim_{z\to\pm(2n-1)\pi/2}\frac{\sin z}{-\sin z}=\lim_{z\to\pm(2n-1)\pi/2}(-1)$$

$$=-1$$

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⑦ State the Cauchy's integral theorem.

Sol:

If $f(z)$ is analytic & $f'(z)$ is continuous at all points inside & on a simple closed curve C then $\int_C f(z)\,dz=0$.

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(8) Find the residue of $f(z) = \tan z$ at $z = \frac{\pi}{2}$.

Sol: Let $f(z) = \frac{\sin z}{\cos z} = \frac{\phi(z)}{\psi(z)}$

$$\operatorname{Res}_{z=a} f(z) = \lim_{z \to a} \frac{\phi(z)}{\psi'(z)}$$

$$\operatorname{Res}_{z=\pi/2} f(z) = \lim_{z \to \pi/2} \frac{\sin z}{-\sin z}$$

$$= \lim_{z \to \pi/2} (-1) = -1$$

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(9) Define & give an example of essential singular points.

Sol:
A point $z=a$ is said to be essential singular point if $z=a$ is singular point & it is not a removable singular point.

Example $f(z) = \sin \frac{1}{z}$.

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(10) Express $\int_0^\pi \frac{d\theta}{2\cos\theta + \sin\theta}$ as complex integration.

Sol: Let $z = e^{i\theta}$

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \; ; \; \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\int_0^\pi \frac{d\theta}{2\cos\theta + \sin\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{2\cos\theta + \sin\theta}$$

$$= \frac{1}{2}\int_C \frac{dz/iz}{\left(z + \frac{1}{z}\right) + \frac{1}{2i}\left(z - \frac{1}{z}\right)} \quad , C \text{ is } |z| = 1$$

$$= \frac{1}{2}\int_C \frac{dz/iz}{\left(\frac{z^2+1}{z}\right) + \left(\frac{z^2-z}{2iz}\right)} = \frac{1}{4}\int_C \frac{dz}{(1+2i)z^2 - z + 2i}$$

DIFFERENTIAL EQUATIONS

① Find the particular integral of $(D-1)^2 y = e^x \sin x$.

Ans: $P.I = \frac{1}{(D-1)^2} e^x \sin x = e^x \frac{1}{[(D+1)-1]^2} \sin x \quad (\because D \to D+a = D+1)$

$= e^x . \frac{1}{D^2} \sin x = e^x \frac{1}{D}(-\cos x) = -e^x \sin x$

② Solve $(D^3 - 3D^2 + 3D - 1)y = 0$

Ans: Auxl. eqn. is $m^3 - 3m^2 + 3m - 1 = 0$

$$\begin{array}{c|cccc} m=1 & 1 & -3 & 3 & -1 \\ & & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & \boxed{0} \end{array}$$

$m^2 - 2m + 1 = 0$

$(m-1)(m-1) = 0$

$\therefore m = 1, 1, 1$

Hence $y = Ae^x + Bxe^x + Cx^2 e^x$

③ If $1 \pm 2i, 1 \pm 2i$ are the roots of the auxiliary eqn. corresponding to a 4th order homogeneous linear differential eqn. $F(D)y = 0$, find its solution.

Ans: Given $m = 1 \pm 2i, 1 \pm 2i$

$\therefore y = e^x(A\cos 2x + B\sin 2x) + xe^x(C\cos 2x + D\sin 2x)$

④ Find the particular integral of the eqn. $(D^2 - 9)y = e^{-3x}$.

Ans: $P.I = \frac{1}{D^2 - 9} e^{-3x} = \frac{x}{2D} e^{-3x} \quad (\because D = a = -3)$

$P.I = \frac{-x}{6} e^{-3x}$

⑤ Find the P.I of $y'' + y' = x^2 + 2x + 4$.

Ans: $P.I = \frac{1}{D^2 + D}(x^2 + 2x + 4) = \frac{1}{D}(1+D)^{-1}(x^2 + 2x + 4)$

$D = 2x + 2$

$D^2 = 2$

$= \frac{1}{D}(1 - D + D^2 - \dots)(x^2 + 2x + 4)$

$= \frac{1}{D}(x^2 + 2x + 4 - 2x - 2 + 2) = \frac{1}{D}(x^2 + 4) = \frac{x^3}{3} + 4x$

⑥ Find the particular integral of $(D^2 - 2D + 1)y = \cosh x$.

Ans: WKT $\cosh x = \frac{e^x + e^{-x}}{2}$

$P.I = \frac{1}{D^2 - 2D + 1}\left(\frac{e^x}{2}\right) + \frac{1}{D^2 - 2D + 1}\left(\frac{e^{-x}}{2}\right)$

$= \frac{x}{2} \frac{1}{2D - 2} e^x + \frac{e^{-x}}{8} = \frac{x^2}{4} e^x + \frac{e^{-x}}{8}$

(7) Find the particular integral of $(D^2-4D+4)y = 2^x$.

Ans: $P.I = \frac{1}{D^2-4D+4} e^{x\log 2} = \frac{1}{(\log 2)^2-4(\log 2)+4} 2^x$

(8) Convert $x^2y''-2xy'+2y=0$ into a linear differential equation with constant coefficients.

Ans: Put $x=e^z \Rightarrow \log x = z$; $xD=D'$, $x^2D^2=D'(D'-1)$

$(D'^2-D'-2D'+2)y=0 \Rightarrow D'^2-3D'+2y=0$

(9) Convert the equation $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=\log x$ into a differential eqn. with constant coefficients.

Ans: Put $x=e^z \Rightarrow \log x = z$; $xD=D'$, $x^2D^2=D'(D'-1)$

$(D'^2-D'-2D'+2)y=z \Rightarrow (D'^2-3D'+2)y=z$

(10) Convert the equation $(2x+3)^2y''-2(2x+3)y'-12y=6x$ as a linear equation with constant coefficients.

Ans: Put $2x+3=e^z \Rightarrow \log(2x+3)=z$ $\quad$ $(2x+3)D=D'$

$(2x+3)^2D^2=D'(D'-1)$

$2x=e^z-3 \Rightarrow x=\frac{1}{2}(e^z-3)$

$\therefore (D'^2-D'-2D'-12)y=\frac{6}{2}(e^z-3) \Rightarrow (D'^2-3D'-12)y=3(e^z-3)$

(11) Reduce the equation $x^4y'''-x^3y''+x^2y'=1$ into linear equation with constant coefficients.

Ans: Given $(x^4D^3-x^3D^2+x^2D)y=1 \Rightarrow (x^3D^3-x^2D^2+xD)y=1/x$

$x^2D^2=D'(D'-1)$, $x^3D^3=D'(D'-1)(D'-2)$

Put $e^z=x \Rightarrow z=\log x$; $xD=D'$,

$((D'^2-D')(D'-2)-(D'^2-D')+D')y=\frac{1}{e^z}$

$(D'^3-2D'^2-D'^2+2D'-D'^2+D'+D')y=e^{-z} \Rightarrow (D'^3-4D'^2+4D')y=e^{-z}$

(12) Find the Wronskian of y_1, y_2 of $y''-2y'+y=e^x\log x$.

Ans: Given $(D^2-2D+1)y=e^x\log x$.

Auxl. eqnl. is $m^2-2m+1=0$

$(m-1)(m-1)=0$

$\therefore m=1,1$

$\therefore$ C.F. is Ae^x+Bxe^x

Here $y_1=e^x$, $y_2=xe^x$; $y_1'=e^x$, $y_2'=xe^x+e^x$

Wronskian $=\begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} = y_1y_2'-y_1'y_2 = xe^{2x}+e^{2x}-xe^{2x}=e^{2x}$

(13) Obtain the differential equation of x alone, given $x' = 7x - y$ & $y' = 3x + y$.

Ans: Given $Dx - 7x + y = 0$, $Dy - 3x - y = 0$

$(D-7)x + y = 0$ —(1) $\quad -3x + (D-1)y = 0$ —(2)

(1) $\times (D-1) \Rightarrow (D-1)(D-7)x + (D-1)y = 0$

$$-3x \qquad + (D-1)y = 0$$
$$(+) \qquad (-)$$

$$(D^2 - D - 7D + 7 + 3)x = 0$$
$$(D^2 - 8D + 10)x = 0$$

(14) Guess the trial solution of the particular integral for the differential equation $y'' + 4y = \cos 2x$ using method of underdetermined coefficients.

Ans: Trial solution for P.I $= c_1 \sin 2x + c_2 \cos 2x$

(15) Convert $(3x^2D^2 + 5xD + 7)y = \frac{2}{x\log x}$ into an equation with constant coefficients.

Ans: Put $x = e^z \Rightarrow \log x = z$; $xD = D'$, $x^2D^2 = D'(D'-1)$

$$\therefore [3(D'^2 - D') + 5D' + 7]y = \frac{2}{e^z . z}$$

$$(3D'^2 - 3D' + 5D' + 7)y = \frac{2}{ze^z}$$

$$(3D'^2 + 2D' + 7)y = \frac{2}{ze^z}$$